附录 1

$$
\begin{aligned}
\nabla_t &= \frac{\partial l_t}{\partial \lambda_t} \\
&= -\frac{1}{2}\left[\frac{\partial \log|F_t|}{\partial \lambda_t^\top} + \frac{\partial v_t^\top F_t^{-1} v_t}{\partial \lambda_t^\top}\right]^\top \\
&= -\frac{1}{2}\left[\frac{1}{|F_t|}\frac{|F_t|}{\partial (vec(F_t))^\top}\frac{\partial vec(F_t)}{\partial \lambda_t^\top} + \frac{\partial v_t^\top F_t^{-1} v_t}{\partial v_t^\top}\frac{\partial v_t}{\partial \lambda_t^\top} + \frac{\partial v_t^\top F_t^{-1} v_t}{\partial [vec(F_t)]^\top}\frac{\partial vec(F_t)}{\partial \lambda_t^\top}\right]^\top \\
&= -\frac{1}{2}\left[\frac{1}{|F_t|}|F_t|[vec((F_t)^{-1})]^\top \dot{F}_t + 2v_t^\top F_t^{-1}\dot{v}_t - \left[\left(F_t^{-1}v_t\right)\otimes\left(F_t^{-1}v_t\right)\right]^\top \dot{F}_t\right]^\top \\
&= -\frac{1}{2}\left[[vec((F_t)^{-1})]^\top \dot{F}_t + 2v_t^\top F_t^{-1}\dot{v}_t - \left(v_t^\top \otimes v_t^\top\right)\left(F_t^{-1}\otimes F_t^{-1}\right)\dot{F}_t\right]^\top \\
&= -\frac{1}{2}\left[\dot{F}_t^{\top} vec((F_t)^{-1}) - \dot{F}_t^{\top}\left(F_t^{-1}\otimes F_t^{-1}\right)(v_t\otimes v_t) + 2\dot{v}_t^{\top} F_t^{-1} v_t\right] \\
&= \frac{1}{2}\left[\dot{F}_t^{\top}\left(F_t^{-1}\otimes F_t^{-1}\right)(v_t\otimes v_t) - \dot{F}_t^{\top} vec[(F_t)^{-1}F_t(F_t)^{-1}] - 2\dot{v}_t^{\top} F_t^{-1} v_t\right] \\
&= \frac{1}{2}\left[\dot{F}_t^{\top}\left(F_t^{-1}\otimes F_t^{-1}\right)(v_t\otimes v_t) - \dot{F}_t^{\top}\left(F_t^{-1}\otimes F_t^{-1}\right)vec(F_t) - 2\dot{v}_t^{\top} F_t^{-1} v_t\right] \\
&= \frac{1}{2}\left[\dot{F}_t^{\top}\left(F_t^{-1}\otimes F_t^{-1}\right)[v_t\otimes v_t - vec(F_t)] - 2\dot{v}_t^{\top} F_t^{-1} v_t\right] \\
&= \frac{1}{2}\left[\dot{F}_t^{\top}\left(F_t^{-1}\otimes F_t^{-1}\right)vec\left(v_t v_t^\top - F_t\right) - 2\dot{v}_t^{\top} F_t^{-1} v_t\right].
\end{aligned}
$$

附录 2

在给定了得分向量∇_t和信息矩阵I_t后，便可以得到时变指数衰减因子λ_t的动态过程。随之而来的一个问题在于，得分向量∇_t和信息矩阵I_t的表达式中均含有未知的动态过程$\dot{v}_t$和$\dot{F}_t$，我们同样在 Kalman 递归表达式的基础上得到它们的动态过程。

根据$v_t = y_t - Z_t a_t$,可得，

$$
\begin{aligned}
\dot{v}_t = \frac{\partial v_t}{\partial \lambda_t} = & \frac{\partial (y_t - Z_t a_t)}{\partial \lambda_t} = -\frac{\partial Z_t a_t}{\partial \lambda_t} \\
= & -\left[\frac{\partial Z_t a_t}{\partial vec(Z_t)^\top}\frac{\partial vec(Z_t)}{\partial \lambda_t} + \frac{\partial Z_t a_t}{\partial a_t^\top}\frac{\partial a_t}{\partial \lambda_t}\right] \\
= & -\left[\left(a_t^\top \otimes I_n\right)\dot{Z}_t + Z_t\dot{a}_t\right].
\end{aligned}
$$

由$F_t = Z_t P_t Z_t^\top + H$ 可得，

$$
\begin{aligned}
\dot{F}_t = & \frac{\partial vec(F_t)}{\partial \lambda_t} = \frac{\partial vec\left(Z_t P_t Z_t^\top\right)}{\partial vec(Z_t)^\top} \frac{\partial vec(Z_t)}{\partial \lambda_t} + \frac{\partial vec\left(Z_t P_t Z_t^\top\right)}{\partial vec(P_t)^\top} \frac{\partial vec(P_t)}{\partial \lambda_t} \\
= & \left(I_{n^2} + \mathcal{C}_{nn}\right)(Z_t P_t \otimes I_n)\dot{Z}_t + (Z_t \otimes Z_t)\dot{P}_t
\end{aligned}
$$

其中，$\dot{P}_t = \frac{\partial vec(P_t)}{\partial \lambda_t}$, $\mathcal{C}_{nn}$表示 $n \times n$ 阶矩阵的交换矩阵(Commutation Matrix)，满足对于一个 $n \times n$ 矩阵 X，有$\mathcal{C}_{nn} vec(X) = vec(X^\top)$.

借助于 Kalman 递归表达式，下面推导$\dot{a}_t$和$\dot{P}_t$的动态。由于$a_{t+1} = \mu + \Phi a_t + K_t v_t$,,可得

$$
\begin{aligned}
\dot{a}_{t+1} = \frac{\partial\, a_{t+1}}{\partial \lambda_t} = & \frac{\partial\, \Phi\, a_t}{\partial a_t^\top} \frac{\partial\, a_t}{\partial \lambda_t} + \frac{\partial\, K_t v_t}{\partial\, vec(K_t)^\top} \frac{\partial\, vec(K_t)}{\partial\, \lambda_t} + \frac{\partial\, K_t v_t}{\partial\, vec(v_t)^\top} \frac{\partial\, vec(v_t)}{\partial\, \lambda_t} \\
= & \Phi \dot{a}_t + \left(v_t^\top \otimes I_3\right)\dot{K}_t + K_t \dot{v}_t
\end{aligned}
$$

其中，$\dot{K}_t = \frac{\partial\, vec(K_t)}{\partial\, \lambda_t}$.

利用卡尔曼增益的表达式$K_t = \Phi P_t Z_t^\top F_t^{-1}$，

$$
\begin{aligned}
\dot{K}_t = & \frac{\partial vec(K_t)}{\partial\, \lambda_t} = \frac{\partial vec\left(\Phi P_t Z_t^\top F_t^{-1}\right)}{\partial\, \lambda_t} \\
= & \frac{\partial vec\left(\Phi P_t Z_t^\top F_t^{-1}\right)}{\partial vec(P_t)^\top} \frac{\partial vec(P_t)}{\partial\, \lambda_t} + \frac{\partial vec\left(\Phi P_t Z_t^\top F_t^{-1}\right)}{\partial vec\left(Z_t^\top\right)^\top} \frac{\partial vec\left(Z_t^\top\right)}{\partial vec(Z_t)^\top} \frac{\partial vec(Z_t)}{\partial\, \lambda_t} \\
& + \frac{\partial vec\left(\Phi P_t Z_t^\top F_t^{-1}\right)}{\partial vec\left(F_t^{-1}\right)^\top} \frac{\partial vec\left(F_t^{-1}\right)}{\partial vec(F_t)^\top} \frac{\partial vec(F_t)}{\partial\, \lambda_t} \\
= & \left(F_t^{-1} Z_t \otimes \Phi\right)\dot{P}_t + \left(F_t^{-1} \otimes \Phi P_t\right)\mathcal{C}_{n,3}\dot{Z}_t - \left(I_n \otimes \Phi P_t Z_t^\top\right)\left(F_t^{-1} \otimes F_t^{-1}\right)\dot{F}_t \\
= & \left(F_t^{-1} Z_t \otimes \Phi\right)\dot{P}_t + \left(F_t^{-1} \otimes \Phi P_t\right)\mathcal{C}_{n,3}\dot{Z}_t - \left(F_t^{-1} \otimes K_t\right)\dot{F}_t
\end{aligned}
$$

根据$P_{t+1} = \Phi P_t \Phi^\top - K_t F_t K_t^\top + Q$，

$$
\begin{aligned}
\dot{P}_{t+1} = & \frac{\partial vec(P_{t+1})}{\partial\, \lambda_t} = \frac{\partial vec(\Phi P_t \Phi^\top)}{\partial\, \lambda_t} - \frac{\partial vec\left(K_t F_t K_t^\top\right)}{\partial\, \lambda_t} \\
= & \frac{\partial vec(\Phi P_t \Phi^\top)}{\partial vec(P_t)^\top} \frac{\partial vec(P_t)}{\partial\, \lambda_t} - \frac{\partial vec\left(K_t F_t K_t^\top\right)}{\partial vec(K_t)^\top} \frac{\partial vec(K_t)}{\partial\, \lambda_t} - \frac{\partial vec\left(K_t F_t K_t^\top\right)}{\partial vec(F_t)^\top} \frac{\partial vec(F_t)}{\partial\, \lambda_t} \\
= & (\Phi \otimes \Phi)\dot{P}_t - \left(I_{3^2} + \mathcal{C}_{3,3}\right)(K_t F_t \otimes I_3)\dot{K}_t - (K_t \otimes K_t)\dot{F}_t
\end{aligned}
$$